ON SUBSPACES OF QUOTIENTS OF $(\Sigma G_n)_{l_p}$ AND $(\Sigma G_n)_{c_0}$

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ABSTRACT

Let $1 < p < \infty$ (resp. $p = \infty$). Then every $\mathscr{L}_p$-subspace of a quotient space of l_p (resp. c_0) is isomorphic to l_p (resp. c_0).

1. Introduction.

In this paper we show that if a subspace X of a quotient of a Banach space of the form $(\Sigma G_n)_{l_p}$ $(1 < p < \infty)$ or $(\Sigma G_n)_{c_0}$ with $\dim G_n < \infty$ admits a nice enough structure, then X is itself isomorphic to a space of the form $(\Sigma E_n)_{l_p}$ or $(\Sigma E_n)_{c_0}$ with $\dim E_n < \infty$. The most significant application of this result is that for $1 < p \leqq \infty$, every $\mathscr{L}_p$ subspace of a quotient of l_p (or c_0 if $p = \infty$) is isomorphic to l_p (or c_0). This answers a question of Lindenstrauss and Pelczynski [7] and improves the result of the second-named author [9] that if X is a $\mathscr{L}_{\infty,\lambda}$ subspace of c_0 with $\lambda < 7/6$, then X is isomorphic to c_0.

Before giving a precise statement of our main result, we mention the kinds of structures that concern us here. A sequence (E_n) of finite dimensional subspaces of the Banach space X is called a finite dimensional decomposition for X (f.d.d., in short) provided that, for each x in X, there is a unique sequence $P_n x \in E_n$ with $x = \Sigma P_n x$. If $(P_n^* X^*)$ forms an f.d.d. for X^*, the decomposition is called shrinking. A f.d.d. (E_n) is an l_p $(1 \leqq p < \infty)$ (respectively, c_0) decomposition provided that, given $e_n \in E_n$, Σe_n converges if and only if $\Sigma \| e_n \|^p < \infty$ (respectively, $\| e_n \| \to 0$).

Our main result is:

THEOREM 1. *Let (G_n) be a sequence of finite dimensional spaces. Suppose (E_n) is a shrinking f.d.d. for a subspace X of a quotient of $(\Sigma G_n)_{l_p}$ $(1 < p < \infty)$*

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{respectively, $(\sum G_n)_{c_0}$}. Then there exist integers $1 = k_1 < k_2 < k_3 \cdots$ so that $([E_i]_{i=k_n}^{k_{n+1}-1})_{n=1}^{\infty}$ is an l_p (respectively, c_0) decomposition for X.

Enflo's remarkable counterexample to the approximation problem [2] points out the futility of attempting to classify even all subspaces of c_0. However, the elementary techniques used here might be refined to classify the isomorphic types of all subspaces of quotients of $(\sum G_n)_{l_p}$ $(1 < p < \infty)$ or $(\sum G_n)_{c_0}$ (dim $G_n < \infty$) which admit a reasonable structure (perhaps only the approximation property) as being of the form $(\sum E_n)_{l_p}$ or $(\sum E_n)_{c_0}$ with $\dim E_n < \infty$. Corollary 2 points in this direction: If X is a subspace of a quotient of $(\sum G_n)_{l_p}$ $(1 < p < \infty)$ or $(\sum G_n)_{c_0}$ $(\dim G_n < \infty)$ and X^* has the approximation property, then X is isomorphic to a complemented subspace of a space of the form $(\sum E_n)_{l_p}$ or $(\sum E_n)_{c_0}$ (with $\dim E_n < \infty$).

Our notation follows that of [5] except that $[A]$ is used instead of span A to denote the closed linear subspace generated by A. Terms used but not defined here are as in [5].

2. The main result

We begin with a basis-theoretic preliminary lemma. Note that $B(X)$ is the closed unit ball of the Banach space X.

LEMMA 1. *Suppose $T: X \to Y$ is a quotient map, (G_n) is an f.d.d. for X, and (E_n) is a shrinking f.d.d. for a subspace of Y. Let a positive integer n and $\varepsilon > 0$ be given. Then there exists a positive integer m so that for any y in $[E_i]_{i=m}^{\infty}$, $\| y \| = 1$, the distance $d(y, T3B[G_i]_{i=n}^{\infty})$ from y to $T3B[G_i]_{i=n}^{\infty}$ does not exceed ε.*

PROOF. Suppose the conclusion is false. Then we may pick $y_m \in [E_i]_{i=m}^{\infty}$ $(m = 1, 2, 3, \cdots)$, $\| y_m \| = 1$, with $d(y_m, T3B[G_i]_{i=n}^{\infty}) > \varepsilon$. By the Hahn-Banach Theorem, there exist $f_m \in Y^*$, $\| f_m \| = 1 = f_m(y_m)$, with $|f_m(Tx)| < 1 - \varepsilon$ for all $x \in 3B[G_i]_{i=n}^{\infty}$. Since (E_i) is shrinking, $y_m \xrightarrow{w} 0$; hence there is a subsequence (z_m) of (y_m) and a corresponding subsequence (g_m) of (f_m) satisfying $|g_m(z_j)| < 1/j$ for $j > m$; $g_m(z_m) = 1$.

Write $z_m = Tw_m$ with $w_m \in X$, $\| w_m \| \leqq 1 + 1/m$. Let $P_i: X \to G_i$ be the obvious coordinate projections. By passing to a subsequence of (w_m), we may assume that $\lim_{m \to \infty} P_i w_m$ exists for each fixed i. Hence by the f.d.d. variant of a well-known result of Bessaga and Pelczynski [1], some subsequence of $(w_{2m} - w_{2m+1})$ is arbitrarily close to a block basic sequence of (G_i). (To apply the result of [1], note that $g_{2m}T(w_{2m} - w_{2m+1}) = g_{2m}(z_{2m}) - g_{2m}(z_{2m+1}) > 1 - \frac{1}{2m+1}$, from which $\| w_{2m} - w_{2m+1} \|$ is bounded away from 0.) Applying this result, we have that there are increasing sequences (p_i) and (k_i) of positive integers and vectors

$$b_m \in [G_i]_{i=k_m}^{k_{m+1}-1}, \| b_m \| \leqq 1 + \frac{1}{2p_m} + \frac{1}{2p_m+1}, \text{ with } \| b_m - (w_{2p_m} - w_{2p_m+1}) \| < \frac{1}{m}.$$

We have $|g_{2p_m}T(w_{2p_m} - w_{2p_m+1})| > 1 - \frac{1}{2p_m} - \frac{1}{2p_m+1}$,

$$\text{and hence } g_{2p_m}(Tb_m) \geqq \left(1 - \frac{1}{2p_m} - \frac{1}{2p_m+1}\right) - \frac{1}{m}.$$

But when $k_m \geqq n$, $b_m \in 3B[G_i]_{i=n}^{\infty}$, so that $|g_{2p_m}(Tb_m)| < 1 - \varepsilon$, which is a contradiction for sufficiently large values of m. Q.E.D.

PROPOSITION 1. *Let (G_n) be a sequence of finite dimensional spaces. Suppose (E_n) is a shrinking f.d.d. for a subspace, X, of a quotient, Y, of $(\sum G_n)_{l_p}$ $(1 < p < \infty)$ or $(\sum G_n)_{c_0}$. Then there is a sequence $1 = p_1 < p_2 < p_3 < \dots$ of integers such that the f.d.d. (X_n) of X defined by $X_n = [E_i]_{i=p_n}^{p_{n+1}-1}$ has the following property: Given any sequence $1 = q(1) < q(2) < q(3) < \cdots$ of integers and $x_m \in [X_i]_{i=q(m)}^{q(m+1)-1}$,*

$$\| \sum x_m \| \leqq \begin{cases} 7(\sum \| x_m \|^p)^{1/p}, & 1 < p < \infty \\ 7(\sup \| x_m \|), & c_0 \text{ case}. \end{cases}$$

PROOF. We treat the case of $(\sum G_n)_{l_p}$; the case of $(\sum G_n)_{c_0}$ requires only notational changes.

Let $T: (\sum G_n)_{l_p} \to Y$ be the quotient map. We select sequences $1 = n_1 < n_2 < n_3 < \cdots$ and $1 = p_1 < p_2 < p_3 < \cdots$ of integers to satisfy for all $1 \leqq i < k$,

(*) for each $x \in [E_j]_{j=p_i}^{p_k-1}$ with $\| x \| = 1$, $d(x, T3B[\sum_{j=n_i}^{n_{k+1}-1} G_j]) < 2^{-i}$;

(**) for each $x \in [E_j]_{j=p_i}^{\infty}$ with $\| x \| = 1$, $d(x, T3B[\sum_{j=n_i}^{\infty} G_j]) < 2^{-i}$.

Set $n_1 = p_1 = 1$, let $n_2 > 1$, and, using Lemma 1, choose $p_2 > p_1$ so that (**) holds for $i = 2$. (**) obviously holds for $i = 1$, so choose $n_3 > n_2$ to make (*) valid for $1 \leqq i < k \leqq 2$. Now select $p_3 > p_2$ so that (**) holds for $i = 3$. Using (**) for $i = 1,2$, choose $n_4 > n_3$ to make (*) true for $1 \leqq i < k \leqq 3$. Continue in this way to define (p_i) and (n_i) by induction.

Set $X_m = [E_i]_{i=p_m}^{p_{m+1}-1}$, let $1 \leqq q(1) < q(2) < \cdots$, and suppose $x_m \in [X_i]_{i=q(m)}^{q(m+1)-1}$.

By (*) there are $w_m \in \sum_{i=n_q(m)}^{n_q(m+1)+1-1} G_i$ with $\| w_m \| \leqq 3 \| x_m \|$ and $\| Tw_m - x_m \| \leqq 2^{-p_q(m)} \| x_m \| \leqq 2^{-m} \| x_m \|$.

But $\| \sum Tw_m \| \leqq \| \sum w_m \| \leqq \| \sum_{m\ odd} w_m \| + \| \sum_{m\ even} w_m \| =$

$$= (\sum_{m\ odd} \| w_m \|^p)^{1/p} + (\sum_{m\ even} \| w_m \|^p)^{1/p} \leqq 2(\sum \| w_m \|^p)^{1/p} \leqq 6(\sum \| x_m \|^p)^{1/p}.$$

Thus $\| \sum x_m \| \leqq 6 (\sum \| x_m \|^p)^{1/p} + \sum 2^{-m} \| x_m \| \leqq 7 (\sum \| x_m \|^p)^{1/p}$. Q.E.D.

PROOF OF THEOREM 1. Let $1 = p_1 < p_2 < \cdots$ and (X_n) be as in the conclusion of Proposition 1. For the case of c_0, simply note that, since (X_n) is an f.d.d., there is a constant c so that, if $x_n \in X_n$, $c \sup \| x_n \| \leqq \| \sum x_n \|$. Thus (X_n) is already a c_0 decomposition.

For the case of $(\sum G_n)_{l_p}$ $(1 < p < \infty)$ we need to dualize Proposition 1. Since a subspace of a quotient is also a quotient of a subspace, it follows that X^* is a subspace of a quotient of $(\sum G_n^*)_{l_q}$ $(1/p + 1/q = 1)$. Also, if P_i is the coordinate projection of X onto X_i, then $(P_i^* X^*)$ is a (necessarily shrinking) f.d.d. for X^*. Thus from Proposition 1, we have that there is a sequence $1 = q(1) < q(2) < \cdots$ of integers so that, if $x_m^* \in [P_i^* X^*]_{i=q(m)}^{q(m+1)-1}$, then $\| \sum x_m^* \| \leqq 7(\sum \| x_m^* \|^q)^{1/q}$ From this it follows that there is a constant c, depending only on the f.d.d. (X_n), so that, if $x_m \in [X_i]_{i=q(m)}^{q(m+1)-1}$, then $c (\sum \| x_m \|^p)^{1/p} \leqq \| \sum x_m \|$. But also $\| \sum x_m \| \leqq 7(\sum \| x_m \|^p)^{1/p}$ by Proposition 1, so $([X_i]_{i=q(m)}^{q(m+1)-1})_{m=1}^{\infty}$ is an l_p decomposition. Q.E.D.

COROLLARY 1. *Let (G_n) be a sequence of finite dimensional spaces, let $1 < p \leqq \infty$, and suppose that X is an $\mathscr{L}_p$ subspace of a quotient of $(\sum G_n)_{l_p}$ (if $1 < p < \infty$) or $(\sum G_n)_{c_0}$ (if $p = \infty$). Then X is isomorphic to l_p (if $1 < p < \infty$) or c_0 (if $p = \infty$).*

PROOF. It is known (cf. [5]) that X admits a shrinking f.d.d.; hence by Theorem 1, X is isomorphic to a space of the form $(\sum E_n)_{l_p}$ (if $1 < p < \infty$) or $(\sum E_n)_{c_0}$ (if $p = \infty$) with $\dim E_n < \infty$. Since X is a $\mathscr{L}_p$ space, it follows easily that there is a constant λ, depending only on X and (E_n), so that, for each n, there is an isomorphism $T_n: E_n \overset{into}{\to} l_p (c_0, \text{if } p = \infty)$ with $\| T_n \| = 1$, $\| T_n^{-1} \| \leqq \lambda$, and there is a projection P_n of l_p (or c_0, if $p = \infty$) onto $T_n E_n$ with $\| P_n \| \leqq \lambda$. From this it follows that X is isomorphic to a complemented subspace of l_p (c_0, if $p = \infty$) and hence by Pelczynski's result [8], X is isomorphic to l_p (c_0, if $p = \infty$). Q.E.D.

REMARK 1. Lindenstrauss [6] showed that l_1 contains a $\mathscr{L}_1$ subspace which is not isomorphic to l_1, so there is no analogue to Corollary 1 for $p = 1$ even for subspaces of l_1.

REMARK 2. In [9] it was shown that, for $1 \leqq \lambda < 7/6$, there is a number $f(\lambda)$ so that if X is a $\mathscr{L}_{\infty,\lambda}$ subspace of c_0, then there is an isomorphism $T: X \overset{onto}{\to} c_0$ with $\| T \| \, \| T^{-1} \| \leqq f(\lambda)$. Further, f depends continuously on λ in the sense that $f(\lambda) \to 1$ as $\lambda \to 1$. This last result does not follow from the techniques in this paper and the corresponding question for $\mathscr{L}_{p,\lambda}$ subspaces of l_p (for $1 < p < \infty$ and even for $p = 1$) remains open.

For subspaces of quotients of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$ which admit a structure weaker than a shrinking f.d.d., we have a weaker version of Theorem 1.

COROLLARY 2. *Let (G_n) be a sequence of finite dimensional spaces and suppose that X is a subspace of a quotient of $(\sum G_n)_{l_p}$ $(1 < p < \infty)$ or $(\sum G_n)_{c_0}$. If X^* has the approximation property, then X is isomorphic to a complemented subspace of $(\sum E_n)_{l_p}$ $(1 < p < \infty)$ or $(\sum E_n)_{c_0}$ for some sequence (E_n) of finite dimensional spaces.*

PROOF. Since X^* is separable, it follows from Grothendieck's results [3] (cf. [5, Remark 4.11]) that X^* has the bounded approximation property. Let (F_n) be a sequence of finite dimensional spaces which is dense (in the Banach-Mazur sense) for the class of all finite dimensional spaces and set $C_p = (\sum F_n)_{l_p}$ $(1 < p < \infty)$, $C_\infty = (\sum F_n)_{c_0}$. C_p is isomorphic to the space C_p discussed in [4]. Now certainly $(\sum G_n)_{l_p}$ $(1 < p < \infty)$ (or $(\sum G_n)_{c_0}$) is complemented in C_p (or C_∞); hence $X \oplus C_p$ is isomorphic to a subspace of a quotient of C_p. But since X^* has the bounded approximation property and is separable, $X \oplus C_p$ has by [4] a shrinking f.d.d. Hence the desired conclusion follows from Theorem 1.

Q.E.D.

REMARK. Continuing in the notation of the proof of Corollary 2, we have from Pelczynski's decomposition method [8] that $X \oplus C_p$ is isomorphic to C_p, since each space is complemented in the other and C_p is isomorphic to $(\sum C_p)_{l_p}$ $\{(\sum C_\infty)_{c_0}$, if $p = \infty\}$. Thus C_p has the property that a Banach space, X, is isomorphic to a complemented subspace of C_p if and only if X is isomorphic to a subspace of a quotient of C_p and X^* has the approximation property.

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